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von Neumann Mutual Information for Anisotropic Coupled Oscillators Interacting with a Single Two-Level Atom

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We consider the interaction between a two-level atom and two electromagnetic fields injected simultaneously within a cavity, with the interaction between the fields in parametric frequency-converter form. The wave function in Schrödinger picture is obtained under certain conditions and consequently the density matrix. By employing a generalization of the von Neumann mutual information (in the context of Tsallis' nonextensive statistics) we measure the degree of entanglement for the present system. An important change is observed in the generalized mutual information depending on the entropic index. We also measure the minimum degree of entanglement during the transition from collapse to revival and vice-versa. Successive revival peaks show a lowering of the local maximum point indicating a dissipative irreversible change in the atomic state.

KEY WORDS: quantum optics; quantum information.

1. INTRODUCTION

Due to the lack of any extensive formalism to deal with a physical system with long-range forces or long-range memory, a generalization of the Boltzmann–Gibbs–Shannon entropy for statistical equilibrium was introduced to the physics world. The main purpose of this generalization is to deal with systems subject to spatial or temporal long-range interactions making their behavior nonextensive. This situation can be seen, for example, in astrophysical environments and plasma physics where the range of interactions is comparable to the size of the system considered. In fact, many relevant mathematical properties of the standard

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thermostatistics are verified by this generalized formalism or can be appropriately generalized. The new entropy has recently been applied to a variety of physical problems, for example to obtain (particular) exact time-dependent solutions for a family of nonlinear Fokker–Planck equations (Eckmann and Ruelle, 1985), where the maximization of what is called Tsallis entropy (under appropriate constraints) established its power. Techniques borrowed from Gibbs–Boltzmann or extensive thermodynamics play an important role in the characterization of complex behavior exhibited by dynamical systems. Analogous notions of entropy, temperature, pressure, and free energy can be applied to quantify the fractal or multifractal attractors of chaotic nonlinear mappings (Beck and Schlogl, 1993; Bohr and Tel, 1988; Tsallis, 1988, 1995). Moreover, one can see a generalized entropy is required to possess the usual properties of positivity, equiprobability, concavity, and irreversibility, but with a suitable extension of the standard additivity for nonextensivity (Tsallis, 2002). However, it is noticed that the new field of quantum information and computation has emerged, not only offering the potential of immense practical computing power, but also suggesting deep links between the well-established disciplines of quantum theory, information theory, and computer science.

Entanglement was found to be a manipulable resource. Under certain conditions, states of low entanglement could be purified into more entangled states by acting locally, and states of higher entanglement could be "diluted" to give larger numbers of less entangled states. A number of entanglement measures have been discussed in the literature, such as the von Neumann reduced entropy, the relative entropy of entanglement (Abdel-Aty, 2000; Abdel-Aty and Abdalla, 2002; Plenio and Vedral, 1998), the so-called entanglement of distillation, and the entanglement of formation (Bennett *et al.*, 1997). Several authors proposed physically motivated postulates to characterize entanglement measures (Abdel-Aty, 2000; Abdel-Aty and Abdalla, 2002; Bennett *et al.*, 1997; Horodecki *et al.*, 2000; Phoenix and Knight, 1988, 1991a,b; Plenio and Vedral, 1998; Vedral *et al.*, 1997). These postulates (although they vary from author to author in the details) have in common that they are based on the concepts of the operational formulation of quantum mechanics (Kraus, 1983). A method using quantum mutual entropy to measure the degree of entanglement in the time development of the Jaynes–Cummings (JC) model has been adopted in Furuichi and Ohya (1999), which we called DEM (degree of entanglement due to mutual entropy). We have formulated the entanglement in the time development of the JC-model with squeezed state (Furuichi and Abdel-Aty, 2001), and then we have shown that the entanglement can be controlled by means of squeezing.

For those reasons we devote the present paper to a rigorous formal derivation of a mathematical expression for the generalized mutual information based on Tsallis entropy, and use that to study the degree of entanglement for the interaction between a two-level atom and two electromagnetic fields injected simultaneously within a perfect cavity, taking into consideration the effect of the interaction between the fields themselves. In this case the Hamiltonian will include the field–field interaction as well as the atom-fields.

It is well known that two types of interaction occur between the fields, frequency conversion and parametric amplification. In the present paper we shall take the interaction to be a parametric frequency conversion. The parametric frequency conversion as it stands is described by a process of exchanging photons between two optical fields of different frequencies, and can be applied to describe various optical phenomena, e.g., to find analogies between frequency conversion and beam splitting (Plastino and Plastino, 1995), or as a lossless linear coupler. In this situation the model is considered to be represented by two electromagnetic waves which are guided inside a structure consisting of two adjacent and parallel waveguides; the linear exchange of energy between these two waveguides is established via the evanescent field. To describe the above system we shall devote the following section to introduce a new Hamiltonian model which consists of three parts; free fields, atom-fields, and field–field, where the solution of the wave function is given. In Section 3 we introduce in brief the generalized von Neumann mutual entropy. Our results and their discussion is given in Section 4, followed by the conclusion in Section 5.

2. ATOM-FIELD HAMILTONIAN

In the framework of the rotating wave approximation we introduce a generalized model Hamiltonian representing the interaction between a two-level atom and two fields injected simultaneously within a perfect cavity. We take the interaction between the fields to be of the parametric down-converter type. The Hamiltonian consists of three parts and can be written as

$$\hat{H} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{FF}} + \hat{H}_{\mathrm{AF}}, \tag{1}$$

where $\hat{H}_{\mathrm{A}}$ is the atomic part of the Hamiltonian and has the form

$$\hat{H}_{\mathrm{A}} = \frac{\hbar\omega_0}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right). \tag{2}$$

The transition frequency between the energy levels for the state $|e\rangle$ and $|g\rangle$ is defined by $\omega_0 = (E_e - E_g)/\hbar$. The other part of the Hamiltonian $\hat{H}_{\mathrm{FF}}$ represents the field–field interaction and has been taken as frequency converter type which has the expression

$$\hat{H}_{\mathrm{FF}} = \hbar\omega_1\hat{a}_1^{\dagger}\hat{a}_1 + \hbar\omega_2\hat{a}_2^{\dagger}\hat{a}_2 + \lambda\hbar\left(\hat{a}_1^{\dagger}\hat{a}_2 + \hat{a}_2^{\dagger}\hat{a}_1\right), \tag{3}$$

where ω_1 and ω_2 are the field frequencies, and λ is the coupling parameter, while $\hat{a}_i$ and $\hat{a}_i^{\dagger}$, are respectively the annihilation and the creation operators for the ith mode of the cavity field satisfying $[\hat{a}_i, \hat{a}_j^{\dagger}] = \delta_{ij}$. The other interaction part of the

Hamiltonian $\hat{H}_{\rm AF}$ is the electric-dipole approximation, which can be written as

$$\hat{H}_{\rm AF} = \hbar\lambda_1(\hat{a}_1^\dagger \otimes |g\rangle\langle e| + \hat{a}_1 \otimes |e\rangle\langle g|) + \hbar\lambda_2(\hat{a}_2^\dagger \otimes |g\rangle\langle e| + \hat{a}_2 \otimes |e\rangle\langle g|), \quad (4)$$

where $|i\rangle\langle j|$, $(i, j = e, g)$ are the atomic pseudospin operators. λ_j, $j = 1, 2$ represent the effective coupling parameters between the atomic system and the field modes. In order to discuss the dynamics of the system we have to find either the solution of Heisenberg equations of motion or to find the explicit expression for the wave function in Schrödinger representation. However, before we proceed further let us introduce the canonical transformation

$$\hat{a}_1 = \hat{b}_1 \cos\xi + \hat{b}_2 \sin\xi, \qquad \hat{a}_2 = \hat{b}_2 \cos\xi - \hat{b}_1 \sin\xi, \quad (5)$$

where $\xi = \frac{1}{2}\tan^{-1}(\frac{2\lambda}{\omega_2-\omega_1})$, and the operators $\hat{b}_i$ and $\hat{b}_j^\dagger$ satisfy the commutation relation $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij} = 1$ if $i = j$ and zero otherwise. It should be noted that the connection between the states $|m_1, m_2\rangle_a$ (say) corresponding to the physical operators $\hat{a}_i$ and $\hat{a}_i^\dagger$, $i = 1, 2$ and the states $|n_1, n_2\rangle_b$ corresponding to the rotated operators $\hat{b}_j$ and $\hat{b}_j^\dagger$, $j = 1, 2$ are given by

$$\begin{aligned}|m_1, m_2\rangle_a = &\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}(-)^j \binom{m_1}{i}\binom{m_2}{j}\left(\frac{n_1!n_2!}{m_1!m_2!}\right)^{\frac{1}{2}} \\ &\times(\cos\xi)^{m_2-j+i}(\sin\xi)^{m_1-i+j}|n_1, n_2\rangle_b, \end{aligned} \quad (6)$$

where $n_1 = i + j$, and $n_2 = m_1 + m_2 - i - j$. In this case one can show that $\hat{a}_i^\dagger\hat{a}_i|m_1, m_2\rangle_a = m_i|m_1, m_2\rangle_a$, $i = 1, 2$ and similarly for the other operators $\hat{b}_j^\dagger\hat{b}_j^\dagger|n_1, n_2\rangle = n_j|n_1, n_2\rangle$, $j = 1, 2$.

After we applied the canonical transformation (5) the Hamiltonian (1) takes the form

$$\begin{aligned}\hat{H} = &\sum_{i=1}^{2}\hbar\Omega_i\hat{b}_i^\dagger\hat{b}_i + \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \hbar\mu_1\big(\hat{b}_1^\dagger|g\rangle\langle e| + \hat{b}_1|e\rangle\langle g|\big) \\ &+\hbar\mu_2\big(\hat{b}_2^\dagger|g\rangle\langle e| + \hat{b}_2|e\rangle\langle g|\big),\end{aligned} \quad (7)$$

where μ_i, $i = 1, 2$ are the modified coupling parameters given by

$$\begin{aligned}\mu_1 &= (\lambda_1\cos\xi - \lambda_2\sin\xi), \\ \mu_2 &= (\lambda_2\cos\xi + \lambda_1\sin\xi),\end{aligned} \quad (8)$$

and Ω_i, $i = 1, 2$ are the new free field frequencies such that

$$\begin{aligned}\Omega_1 &= (\omega_1\cos^2\xi + \omega_2\sin^2\xi - \lambda\sin 2\xi), \\ \Omega_2 &= (\omega_2\cos^2\xi + \omega_1\sin^2\xi + \lambda\sin 2\xi).\end{aligned} \quad (9)$$

In the interaction picture the Schrödinger equation can be written as follows

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = V^{\mathrm{I}}(t)|\psi(t)\rangle, \tag{10}$$

where $V^{\mathrm{I}}(t)$ is the interaction term which is of the form

$$\begin{aligned} V^{\mathrm{I}}(t) = \mu_1\big[\hat{b}_1^\dagger \otimes \hat{\sigma}_- e^{i\Delta_1 t} + \hat{b}_1 \otimes \hat{\sigma}_+ e^{-i\Delta_1 t}\big] \\ + \mu_2\big[\hat{b}_2^\dagger \otimes \hat{\sigma}_- e^{i\Delta_2 t} + \hat{b}_2 \otimes \hat{\sigma}_+ e^{-i\Delta_2 t}\big], \end{aligned} \tag{11}$$

with the new detuning parameters $\Delta_j = (\Omega_j - \omega_0), \quad j = 1, 2$.

Since the interaction Hamiltonian after we apply the canonical transformation is given in terms of the rotating operators, we have to write the state $|\psi(t)\rangle$ in terms of the number states belonging to the same operators. However, due to the difficulty in solving the system of equations resultant of the Schrödinger equation, we shall adjust the coupling parameter λ to take the form

$$\lambda = \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)}\varepsilon, \qquad \text{where} \qquad \varepsilon = \frac{\omega_2-\omega_1}{\lambda_2-\lambda_1}, \tag{12}$$

In this case the restrictive condition (12) implies that the coupling parameter μ_1 tends to zero while the coupling parameter μ_2 survives and equals $\eta = \sqrt{\lambda_1^2+\lambda_2^2}$. Here we may point out that the operator $\hat{N}_1 = \hat{b}_1^\dagger\hat{b}_1$ is a constant of the motion and this in fact allows us to find the explicit solution of the wave function in the present case. This can be seen from Eq. (8) as a consequence of assuming the coupling parameter $\mu_1 = 0$. It also interesting to point out that if the field frequencies $\omega_1 = \omega_2$ then the system will reduce automatically to the isotropic case, for more details see Abdalla *et al.* (2002). Consequently we can write after some manipulations the wave function $|\psi(t)\rangle$ as follows

$$\begin{aligned} |\psi(t)\rangle = & \sum_{n_1,n_2=0}^{\infty}\Bigg[\cos\frac{\theta}{2}\left\{\cos g_{n_2+1}t - i\frac{\Delta_2}{2}\frac{\sin g_{n_2+1}t}{g_{n_2+1}}\right\}q_{n_1,n_2} \\ & -ie^{-i\phi}\sin\frac{\theta}{2}\frac{\sin g_{n_2+1}t}{g_{n_2+1}}\{\eta\sqrt{n_2+1}q_{n_1,n_2+1}\}\Bigg]|n_1,n_2\rangle_b\otimes|e\rangle \\ & +\sum_{n_1,n_2=0}^{\infty}\Bigg[e^{-i\phi}\sin\frac{\theta}{2}\left\{\cos g_{n_2}t + i\frac{\Delta_2}{2}\frac{\sin g_{n_2}t}{g_{n_2}}\right\}q_{n_1,n_2} \\ & -i\cos\frac{\theta}{2}\frac{\sin g_{n_2}t}{g_{n_2}}\{\eta\sqrt{n_2}q_{n_1,n_2-1}\}\Bigg]|n_1,n_2\rangle_b\otimes|g\rangle. \\ = & \sum_{n_1,n_2=0}^{\infty}[A(n_1,n_2,t)|n_1,n_2\rangle_b\otimes|e\rangle + B(n_1,n_2,t)|n_1,n_2\rangle_b\otimes|g\rangle] \end{aligned} \tag{13}$$

where $g_{n_2} = \sqrt{\frac{\Delta_2^2}{4} + \eta^2 n_2}$. Here we may point out that in Eq. (13) the combination between the excited state and the ground state of the atom has been taken initially in the form

$$|\psi(0)\rangle = \sum_{n_1,n_2=0}^{\infty} q_{n_1,n_2}\big[\cos\frac{\theta}{2}|e\rangle + \sin\frac{\theta}{2}e^{-i\phi}|g\rangle\big] \otimes |n_1, n_2\rangle_b], \tag{14}$$

therefore to reach the excited states we have to take $\theta = 0$ while the ground states correspond to $\theta = \pi$.

It would be interesting to say that, in the present case of the two modes the time evolution of the wave function depends on the initial photon number distribution especially in the rotated modes as they appear in the last equation.

In what follows we consider the two modes of the field in the rotated bases to be uncorrelated coherent state $|\beta_1, \beta_2\rangle = |\beta_1\rangle \otimes |\beta_2\rangle$ such that

$$\begin{aligned}|\beta_1, \beta_2\rangle &= \exp\left(-\frac{1}{2}\big[|\beta_1|^2 + |\beta_2|^2\big]\right) \sum_{n_1,n_2=0}^{\infty} \frac{\beta_1^{n_1}\beta_2^{n_2}}{\sqrt{n_1!n_2!}}|n_1, n_2\rangle_b \\ &= \sum_{n_1,n_2=0}^{\infty} q_{n_1,n_2}|n_1, n_2\rangle_b,\end{aligned} \tag{15}$$

and therefore the quantity q_{n_1,n_2} in Eq. (13) represents the amplitude of the state $|n_j\rangle_b$ of the jth mode. It takes the form $q_{n_j} = (\beta_j^n/\sqrt{n_j!})\exp(-\bar{n}_j/2)$ with $\beta_j = \sqrt{\bar{n}_j}\exp(i\zeta_j)$, $\bar{n}_j$ and ζ_j represent the initial average photon number and the phase angle of the excitation for j $(j \equiv 1, 2)$ mode, respectively, in a coherent state.

For such coherent states it is easy to establish that

$$\beta_1 = \alpha_1\cos\xi - \alpha_2\sin\xi, \qquad \text{and} \qquad \beta_2 = \alpha_2\cos\xi + \alpha_1\sin\xi, \tag{16}$$

where α_i, is the eigenvalue of the physical operators $\hat{a}_i$, $i = 1, 2$ with respect to the coherent states $|\alpha_i\rangle$.

The Eq. (16) governs the relation between the photon states in the original (physical) and the rotated (artifical) states. Therefore, if we use this relation, then one can transform from one set of bases to another. When the two modes are uncorrelated then q_{n_1,n_2} in Eq. (15) can be factored into two modes of which one depends on n_1 and the other depends on n_2. Hence the summation over n_1 factors is out because the argument of the sinusoidal function depends on n_2 only and we are left with a modified JCM with vacuum Rabi frequency $\eta = \sqrt{\lambda_1^2 + \lambda_2^2}$ and the effective mode in this case is the b_2 mode.

Although the expression of the wave function in the present case is similar to that of the isotropic case, see Abdalla *et al.* (2002), however there is a main difference between the two cases, that is; the detuning parameter Δ_2 includes the ratio between the two coupling parameters λ_1, λ_2 and in addition it includes the frequencies of the two fields and the atom. Thus, in our computations we have not to ignore this fact. Having obtained the explicit form of the wave function, we are therefore in a position to discuss the statistical properties of the system. Our main purpose is to study the effects of all parameters on the degree of entanglement, where we shall take into consideration strong and weak coupling regimes. This will be seen in the following section.

3. GENERALIZED VON NEUMANN MUTUAL ENTROPY

Since we are concerned with studying the effect of the generalized von Neumann mutual entropy on the Hamiltonian system (1), therefore, let us first introduce some essential concepts related to it. Suppose N be a positive constant integer, and let us denote by $I_N = \{1, 2, \ldots, N\}$ the set of integers from 1 to N, and suppose we define the set of all probability distributions on I_N by

$$A_N = \left\{ (p_1, \ldots, p_N) \in \Re, \quad 0 \leq p_i \leq 1 \quad \forall \quad i \in I_N, \quad \text{and} \quad \sum_{i=1}^{N} p_i = 1 \right\}. \tag{17}$$

Then Tsallis' thermostatistics is recognized as a new paradigm for statistical mechanical considerations. One of its crucial ingredients, Tsallis' normalized probability distribution is obtained (Tsallis *et al.*, 1998) by the well known MaxEnt definition (Jaynes, 1963; Katz, 1967). Now let us define the function S_q(the generalized entropy) on A_N by

$$S_q = \begin{cases} -\frac{k}{q-1}\left(1 - \sum\limits_{i=1}^{n} p_i^q\right), & q \neq 1 \\ \\ -k \sum\limits_{i=1}^{n} p_i \ln p_i, & q = 1 \end{cases} \tag{18}$$

where $q \in \Re$, $q > 0$, and k a positive constant. The index q is a parameter unknown a priori and widely believed to be fixed by dynamical details beyond thermodynamical feature of the systems. Therefore, if we denote by $O_j^{(i)}(j = 1, \ldots, n)$ the n relevant observables (Fick and Sauerman, 1990), then the generalized expectation values $\langle\langle O_j \rangle\rangle_q$, are given by

$$\langle\langle O_j \rangle\rangle_q = \frac{\sum_{i=1}^{n} p_i^q O_j^{(i)}}{\sum_{i=1}^{n} p_i^q}. \tag{19}$$

In the case of a continuous probability distribution on $\Re$ with density ρ, the definition given by (18) can be written as

$$S_q = \begin{cases} -\frac{k}{q-1}\int \left\{1-(\rho(x))^{q-1}\right\}\rho(x)dx, & q \neq 1 \\ -\int \rho(x)\ln\rho(x)dx, & q = 1 \end{cases} . \tag{20}$$

Thus we can regard the ordinary statistical mechanics as a special case of the generalized formalism. From the experimental work there is growing evidence that $q \neq 1$ yields a correct description of many complex physical phenomena, including for example hydrodynamic turbulence (Arimitsu and Arimitsu, 2000; Beck, 2000a), scattering processes in particle physics (Beck, 2000b; Bediaga *et al.*, 2000), and self-gravitating systems in astrophysics (Lavagno *et al.*, 1998; Plastino and Plastino, 1993). Based on the entropy Eq. (20), a wealth of papers (see for example (http://tsallis.cat.cbpf.br/biblio.htm) as an updated list) have been presented developing an alternative thermodynamical formalism and applying it to actual physical systems. The central object of information theory, the entropy,which has been introduced in quantum mechanics by von Neumann (Ohya, 1983)

$$S(\hat{\rho}) = -Tr\hat{\rho}\ln\hat{\rho}, \tag{21}$$

where $\hat{\rho}$ is a density matrix. Its relationship to the Shannon entropy $H(\hat{\rho}) = -\Sigma p(X = x_i)\ln p(X = x_i)$, as our measure of the information contained in a random variable X governed by probability distribution p. The above equations become obvious when considering the von Neumann entropy of a mixture of orthogonal states. In this case, the density matrix $\hat{\rho}$ contains classical probabilities p_i on its diagonal, and $S(\hat{\rho}) = H(\hat{\rho})$. In general, however, quantum mechanical density matrices have off-diagonal terms, which, for pure states, reflect the relative quantum phase in superpositions. In classical statistical physics, the concept of conditional and joint probabilities has given rise to the definition of conditional and joint entropies. If $\hat{\rho}$ describes a pure state, then the entropy tends to zero, and if $\hat{\rho}$ describes a mixed state, then the entropy does not equal to zero. Consider F and A that interact with each other. How are the entropies of these systems related to the entropy of the composite system that comprises them both?. The answer to this question was listed by the Araki-Lieb theorem (Araki and Lieb, 1970). Let S_{F} and S_{A} denote the entropies of the two interacting systems and let S be the entropy of the composite system. Araki and Lieb showed that these entropies satisfy the "triangle inequalities"

$$|S_{\mathrm{A}} - S_{\mathrm{F}}| \leq S \leq S_{\mathrm{A}} + S_{\mathrm{F}}. \tag{22}$$

Quantum entropies are generally difficult to compute because they involve the diagonalization of large density matrices (in many cases, infinite dimensional). Thus explicit illustrations of the inequalities Eq. (22) are difficult to come by.

Phoenix and Knight (1988, 1991a,b) gave a nice illustration of these inequalities in the context of the JC model. They considered a two-level atom interacting with an undamped cavity initially in a coherent state. In this case the composite entropy is initially zero and remains zero at all times because the atom–field system is isolated from its environment. Under those circumstances the latter inequality $S \leq S_A + S_F$ is trivially satisfied whereas the former implies that $S_F = S_A$. It is easy to calculate the atomic entropy S_A but the calculation of the field entropy S_F is more problematic. However, Bediaga *et al.* (2000) succeeded in evaluating the field entropy in closed from and showed that it did indeed equal the atomic entropy at all times. The entropies of the atom and the field, when treated as a separate system, are defined through the corresponding reduced density operators by

$$S_{A(F)} = -Tr_{A(F)}\{\hat{\rho}_{A(F)} \ln \hat{\rho}_{A(F)}\}. \tag{23}$$

The field entropy as a measurement of the degree of entanglement between the field and the atom of different systems has been used (see, for example, Abdel-Aty, 2003; Abdel-Aty *et al.*, 2002; Abdel-Aty and Furuichi, 2002). Also, the time development of the entangled state in the JCM has been studied by applying entanglement degree due to (quasi-) mutual entropy which is a special case of the quantum relative entropy type measure (Furuichi and Abdel-Aty, 2001; Furuichi and Ohya, 1999).

A quantity also used to compare distributions as well as quantum states is the mutual information or mutual entropy (Arimitsu and Arimitsu, 2000; Beck, 2000a). The quantum (von Neumann) mutual information $S(\rho)$ relative to two subsystems (A and F) may be written as

$$S(\rho) = S(\rho_A) + S(\rho_F) - S(\rho_{AF}) \tag{24}$$

where $S(\rho_A)$, $(S(\rho_F))$ is the entropy relative to the subsystem A(F), and $S(\rho_{AF})$ is the entropy of the overall state, described by a density operator ρ_{AF}. The reduced density operators relative to the subsystems, ρ_A and ρ_F are obtained from ρ_{AF} through the usual partial tracing operation, $\rho_{A(F)} = Tr_{F(A)}\rho_{AF}$. We may write

$$\hat{\rho}_F(t) = |C(t)\rangle\langle C(t)| + |S(t)\rangle\langle S(t)|, \tag{25}$$

where the bimodal field states $|C(t)\rangle$,and $|S(t)\rangle$, are given by

$$|C(t)\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A(n_1, n_2, t)|n_1, n_2\rangle, \quad |S(t)\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} B(n_1, n_2, t)|n_1, n_2\rangle. \tag{26}$$

The generalization of the von Neumann mutual information based on Tsallis entropy can be written as,

$$\Upsilon_{\text{Tsa.}} = S(\rho_{qA}) + S(\rho_{qF}) - S(\rho_{qAF}). \tag{27}$$

This quantity would represent a generalization of the measure of correlations for a wider class of quantum systems. Next, we construct the most general composable entropy which is reduced into a function of two subsystem Tsallis entropies $S(\rho_{q\mathrm{A}})$ and $S(\rho_{q\mathrm{F}})$ defined as (Vidiella-Barranco, 1999)

$$\begin{aligned}\Upsilon_{\mathrm{Tsa.}} = {} & \frac{1}{1-q}\left[1+\left[\frac{\langle C(t)|C(t)\rangle+\langle S(t)|S(t)\rangle}{2}+\frac{1}{2}\left\{(\langle C(t)|C(t)\rangle-\langle S(t)|S(t)\rangle)^2\right.\right.\right.\\ & \left.\left.\left.+4|\langle C(t)|S(t)\rangle|^2)\right\}^{1/2}\right]^q\right]+\frac{1}{1-q}\left[\frac{\langle C(t)|C(t)\rangle+\langle S(t)|S(t)\rangle}{2}\right.\\ & \left.-\frac{1}{2}\left\{(\langle C(t)|C(t)\rangle-\langle S(t)|S(t)\rangle)^2+4|\langle C(t)|S(t)\rangle|^2\right\}^{1/2}\right]^q\\ & -\frac{2}{1-q}\left\{\langle C(t)|C(t)\rangle^q+\langle S(t)|S(t)\rangle^q\right\}, \end{aligned} \tag{28}$$

In the above equation, the generalized mutual information has been given as a function of the entropic parameter q. In the case of $q = 1$ we get the usual von Neumann mutual information. In the next section we shall discuss the dynamical behavior of the generalized mutual information entropy of the present model based on Tsallis entropy.

4. DISCUSSION OF THE RESULTS

In the present section we shall examine and discuss the effect of the variation of the entropic index q on the generalized mutual information entropy related to the system based on the Hamiltonian given by Eq. (1). As initial condition and for all our plots we have taken the coherence parameter α_i to be real, where its square represents the intensity of the initial coherent field. In the mean time our numerical results have been taken for different values of the involved parameters, and with a great precision an excellent accuracy for the behavior of the generalized mutual information entropy function $\Upsilon_{\mathrm{Tsa.}}$ has been determined.

As a result of the constraint we have imposed on the system and to avoid any violation may occur when $\lambda_1 \to \lambda_2$ we have to restrict our discussion to the case in which the coupling parameters $\lambda_1 \neq \lambda_2$. This means we shall consider two cases; one when $\lambda_2/\lambda_1 \gg 1$ or $\lambda_1/\lambda_2 \gg 1$ (strong coupling case) while the other case is when $(\omega_2 - \omega_1) \gg \eta = \sqrt{\lambda_1^2 + \lambda_2^2}$ (weak coupling case). However the special case in which $\lambda_1 \to \lambda_2$ can only be considered if one takes $\omega_1 \to \omega_2$ in the same time. In Fig. 1, we plot the generalized mutual information entropy $\Upsilon_{\mathrm{Tsa.}}$ given by Eq. (28) as a function of the scaled time ηt. We have assumed that $\theta = \pi/3$ (corresponding to the coherent atomic state), $\phi = 0$ and the field in the coherent

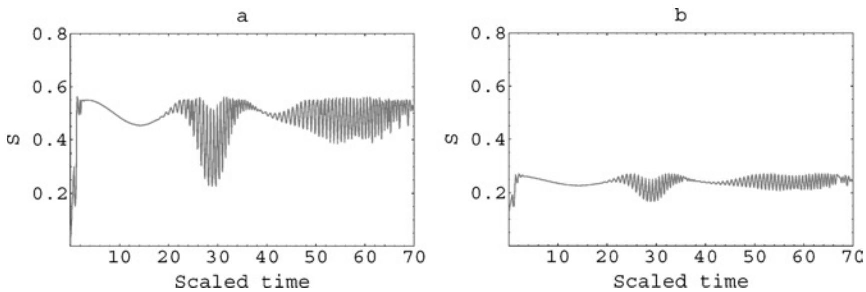

Fig. 1. The evolution of the generalized mutual information as a function of the scaled time ηt. Calculations assume that $\theta = \pi/3, \phi = 0$ and the field in the coherent state with $\alpha_1 = 5, \alpha_2 = 1$, $\omega_1/\eta = 0.5$, $\omega_2/\eta = 1$, the detuning parameter ($\delta = \omega_2 - \omega_0 = 0$), $\lambda_2/\lambda_1 = 1.2$, and for different values of q where (a) $q = 2$, (b) $q = 5$.

states with $\alpha_1 = 5, \alpha_2 = 1$. Moreover let us rewrite the detuning parameter Δ_2 as

$$\Delta_2 = \delta_1 \sin^2 \xi + \delta_2 \cos^2 \xi + \lambda \sin 2\xi, \tag{29}$$

where $\delta_i, i = 1, 2$ are partial detuning parameters defined by $\delta_i = \omega_i - \omega_0$. If we assume $\omega_2 = \omega_0$ corresponding to the exact resonance between the second mode and the atom, then the detuning parameter Δ_2 in this case is equal $(\omega_1 - \omega_0)\lambda_2^2$. This means that the Rabi frequency and consequently the behavior of the generalized mutual information entropy will still be affected by the frequencies of one of the fields and the atom. For fixed values of the field frequencies $\omega_1 = 0.5$ and $\omega_2 = 1$ and for the case in which $\lambda_2/\lambda_1 = 1.2$ we can easily realize in general that as we increase the value of the parameter q (entropic index) there is a decrease in the value of the entropy and consequently we should expect weak entanglement between the fields and the atom. This phenomenon may be compared with the atom–field interaction in the presence of the Kerr-like medium where we can see for small values of the Kerr-like medium, there is an increase of the sustainment time of the maximum field entropy and strong entanglement of the field with the atom, while for large values, it results in a decrease of the field entropy, and the field is disentangled from the atom during the time evolution (Abdel-Aty, 2000; Abdel-Aty and Abdalla, 2002; Plenio and Vedral, 1998). However, in the present case it is also noted that the behavior of the function is quite different to that of the case of the Kerr-like medium. For example, we can observe increment in the value of the entropy after onset of the interaction but for short period of the time, followed by a rapid fluctuation with a strong collapse behavior compared to the rest of the interaction time, see Fig. 1. However for the Kerr-like medium case one can see decrease in the entropy after onset of the interaction. Also, the amplitude in the present case is smaller than that for the case of the Kerr-like medium. Further increase in the value of the entropic index leads to more

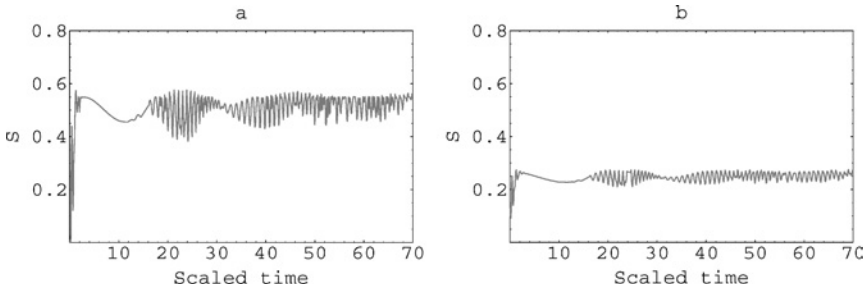

Fig. 2. The same as in Fig. 1 but $\lambda_2/\lambda_1 = 2$.

decrease in the maximum value of the generalized mutual information entropy. While decrease in the ratio value of the coupling parameters $\lambda_2/\lambda_1 = 0.7$ leads to decrease in the interference between the fluctuations where more revival periods can be realized, see Fig. 2. On the other hand, if we increase the ratio of the coupling parameters then the situation lightly changes, see Fig. 3. For example when we take $\lambda_2/\lambda_1 = 2$ then we find for the case in which the entropic index $q = 2$ the generalized mutual information entropy reaches its maximum value similar to the previous case, but the amplitude of the oscillations becomes smaller. This indicates that when the coupling parameter λ_2 is twice the coupling parameter λ_1 the entanglement between the fields and the atom gets more stronger than that of the previous case but just for the particular value of the entropic index $q = 2$ and for a certain period of the time. This behavior has been observed for the other values of $q > 2$, however the function never attained its maximum value again, but it tends asymptotically to zero as q increases.

When we have examined the weak coupling case $(\omega_2 - \omega_1) \gg \eta$ (which is not displayed here) the system shows disentanglement immediately after an increase in the value of the entropic index $q > 1$. Further computations (see Fig. 4) show that as soon as we decrease the value of one of the mean photon numbers such

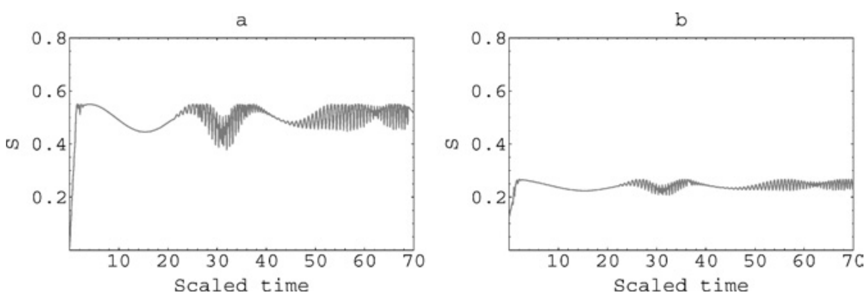

Fig. 3. The same as in Fig. 1 but $\lambda_2/\lambda_1 = 0.7$.

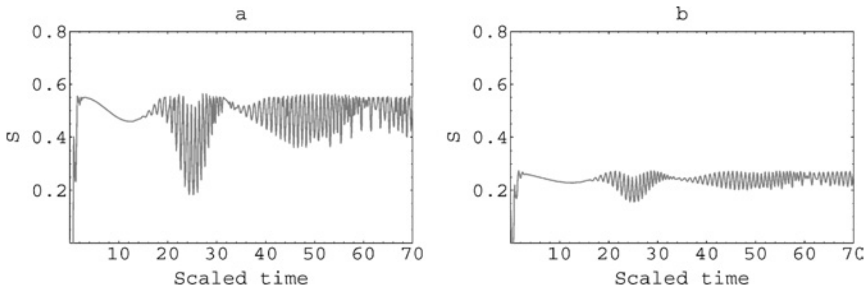

Fig. 4. The same as in Fig. 1 but $\alpha_1 = 5, \alpha_2 = 0.01$.

that $\alpha_1 = 5, \alpha_2 = 0.01$ and without making any changes in the values of the other parameters but taking into consideration the ratio of the coupling parameters to be $\lambda_2/\lambda_1 = 1.2$ we find that the amplitudes of the fluctuations are increased and the phenomenon of collapses which has appeared in the previous two cases washed out while the revivals become more pronounced. The effect of the entropy index parameter is also obvious in this case too, where we can see decrease in the entropy value as q increases.

5. CONCLUSION

In the above sections of the present paper we have introduced a new model of Hamiltonian. This model represents the interaction between a two-level atom and two electromagnetic fields injected simultaneously within a cavity. The interaction between the fields themselves have been taken into consideration. Under a certain integrability condition, exact expression for the wave function is obtained. Based on the wave function we have employed a generalization of the quantum mechanical von Neumann's mutual information within Tsallis' nonextensive statistics to examine the effect of the entropic index on the degree of entanglement. The observable-independent quantity, here denoted as $S_{\mathrm{Tsa.}}$, is important for determining the degree of entanglement between different subsystems. This work can be regarded as a first attempt to establish a connection between an intrinsic property of physical systems and the measure of the degree of entanglement.

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